Dynamics of some neural network models with delay

Jiong Ruan, Lijuan Li, and Wei Lin
Research Center for Nonlinear Science, Laboratory of Mathematics for Nonlinear Science, and Department of Mathematics, Fudan University, People's Republic of China, 200433

(Received 6 June 2000; revised manuscript received 14 December 2000; published 18 April 2001)

The dynamics of the neuronic model described by the one-dimensional delay functional differential equation are studied in this paper. We give a strict and detailed analysis of dynamical characteristic of this model by the Lyapunov functional approach and Hopf bifurcation proposition. Furthermore, numerical simulations, as well as Lyapunov exponents, are presented to support our conjectures about the appearance of complex dynamics such as chaos. We also investigate the dynamics of the neural network model described by the n-dimensional delay functional differential equation with a symmetrical weight matrix, and corresponding simulation results are included as concrete examples.

DOI: 10.1103/PhysRevE.63.051906 PACS number(s): 87.18.Sn

I. INTRODUCTION

Various research on the dynamics of neural network systems has been creating great interest for more than 30 years. In the last ten years, many researchers have focused on the study of the dynamics of neural network models with delay. In fact, neural networks often have time delay. An example of neural network delay is the delay due to the finite switching speed of amplifiers in electronic neural networks. Research on these delay phenomena has been vast and the results are consistent. Marcus and Westervelt [1] incorporated a single time delay into the connection of terms of Hopfield's model and observed sustained oscillations resulting from this time delay. Further investigation on dynamics of such models can be found in Refs. [2–8] and references therein; however, their primary concern is the equilibrium and stability of these neural network systems given specific parameters. Meanwhile, other researchers, such as Pakdaman *et al.* [9] and Brunel and Hakim [10], devoted their research to these oscillations, which are ubiquitous in neural networks. In this paper, we study both stable dynamics and oscillations, even chaos, in the given neural network model with delay.

In 1997, Pakdaman, *et al.* [2,9] considered the following one-dimensional system,

$$\frac{dx(t)}{dt} = -x(t) + A\tanh[x(t-\tau)], \tag{1}$$

and its solution is defined over the infinite-dimensional phase space $S=C[-\tau,0]$ of continuous functions on the interval $[-\tau,0]$. They concluded that for certain parameters, the equation has a stable convergent solution or an oscillating periodic solution, but no chaotic dynamics.

Furthermore, one of the mathematical models, originally proposed by Caianiello and Lucas [11] 30 years ago, consists of a set of neuronic and mnemonic equations. K. Gopalsamy and I. K. C. Leung (1997) [3] investigated the dynamical characteristics of a firing neuron, adjusted using the difference of its current status and the weighted average of its firing history. After appropriate transformations, the neuronic equation changes into the following delay differential equation:

$$\frac{dx(t)}{dt} = -x(t) + A\tanh[x(t)] + B\tanh[x(t-\tau)]. \tag{2}$$

Obviously, the neuron exerts instantaneous self-excitation with the parameter $A>0$ and delayed self-inhibition with $B<0$. Therefore, the signs of the parameters A and B determine the self-characteristic of a neuron with threshold effect. It is reported in Ref. [4] that with the parameters $A>0$, $B<0$ (that is, the neuron is self-excited and its delayed effect is self-inhibitory), the system has global asymptotic stability independent of the delay τ when A and B are in the triangle determined by three lines $A=0$, $B=0$, and $A-B=1$ in the $A-B$ parameter plane.

Since studying the dynamics (not only the stable but the oscillating characteristic as well) of Eq. (2) is essential for understanding the behavior of larger neural networks in the presence of delays, parameters in other regions were examined by Pakdaman and Malta (1998) and the corresponding results [4] are as follows.

When $A+B\leqslant 1$, the origin is the single equilibrium of Eq. (2), while when $A+B>1$, Eq. (2) has three equilibria, which are $x_1=a$, $x_2=0$, and $x_3=-a$, where a is the unique strictly positive real number that satisfies $a=(A+B)\tanh(a)$.

(1) If $A+B<1$ and $A-B<1$ (region Ia), the origin is globally asymptotically stable for all $\tau\geqslant 0$.

(2) If $A+B<1$ and $A-B>1$ (region Ib), the origin loses its stability through a Hopf bifurcation at

$$\tau_c = \cos^{-1}((1-A)/B)/\sqrt{B^2-(1-A)^2}$$

with the delay being increased. For delays larger than τ_c, undamped oscillations appear.

(3) If $A+B>1$, $B>0$ (IIa), $x_1=a$ and $x_3=-a$ are locally asymptotically stable, while x_2 is unstable. For small delay τ, Eq. (2) is convergent in region IIa as the boundary separating the two basins is exactly the stable manifold of the origin and as the delay is increased, the origin undergoes successive Hopf bifurcation leading to the generation of periodic orbits.

(4) If $A+B>1$, and $B<0$ (IIb), the instability is similar to that in region Ib. For large delays, most solutions display asymptotically periodic oscillations.

In the following section, we give a thorough and comprehensive investigation about a one-dimension delay differential equation and establish a good foundation for the numerical simulation; we strictly present proofs about the theoretical results by a Lyapunov functional approach and Hopf bifurcation proposition with our own technique. These mathematical proofs conversely suggest our conjecture about the appearance of the complex dynamics such as chaos. In order to make our result more complete, numerical simulations and Lyapunov exponent calculations are given. In the last section, we analyze the dynamics in a specified class of higher-dimensional neural network systems.

II. DYNAMICS OF THE NEURONIC MODEL DESCRIBED IN A ONE-DIMENSIONAL DELAY FUNCTIONAL DIFFERENTIAL EQUATION

In this section, we give a theoretical analysis of the dynamics of the neuronic model with the parameters A and B in different regions Ia, Ib, IIa, and IIb. The different parameter sets are relevant to self-excitation or self-inhibition of the feedback of a neuron and its threshold effect. In order to present our results, we should first introduce some basic theory of a functional differential equation. Consider the autonomous equation

$$\frac{dx(t)}{dt}=f(x_t),$$

where $f:C\to R(C=C[-\tau,0])$ is completely continuous and solutions of this equation depend continuously upon the initial data. We denote by $x(\phi)$ the solution through $(0,\phi)$. As we have finished defining the solution of the above equation, the asymptotic stability criterion and some useful facts can be presented as follows.

Fact 1 [12]:

(1) Suppose $V:C\to R$ is continuous, $V(0)=0$, and there exist non-negative functions $\mu(r)$ and $\omega(r)$ such that when $r\to\infty$, $\mu(r)\to\infty$, and $\mu(|\phi(0)|)\leqslant V(\phi)$, $\dot{V}(\phi)\leqslant -\omega(|\phi(0)|)$. Then the zero solution of equation $dx(t)/dt=f(x_t)$ is stable and every solution is bounded. If $\omega(r)$ is positive definite, then every solution tends toward zero as $t\to\infty$.

(2) Suppose V is continuous on $\bar{G}$ (the closure of G), $\dot{V}\leqslant 0$ on G, and $x_t(\phi)$ is a bounded solution of Eq. (2), which remains in G. Then $x_t(\phi)\to M$ as $t\to\infty$. Here M denotes the largest invariant set in $S=\{\phi|\phi\in\bar{G},\dot{V}(\phi)=0\}$ with respect to Eq. (2).

Fact 2 [12]: If Eq. (2) satisfies the following conditions, then Eq. (2) has a nonconstant period solution $2\pi/v_0$.

(a) The linear equation has a simple purely imaginary characteristic root $\lambda_0=iv_0\neq 0$ and all characteristic roots $\lambda_j\neq\lambda_0,\bar{\lambda}_0$ satisfy $\lambda_j\neq m\lambda_0$ for any integer m;

(b) $\mathrm{Re}\,\lambda'(0)\neq 0$.

Fact 3 [13]: If

$$\dot{x}(t)=ax(t)+bx(t-\tau)+\sum_{i+j\geqslant 2}c_{ij}(t)x^i(t)x^j(t-\tau)\quad(3)$$

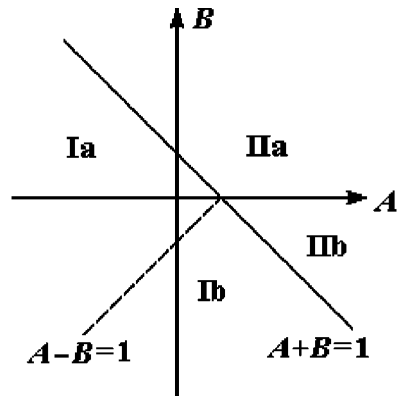

FIG. 1. A-B plane.

satisfies the following conditions, then for any $\tau>0$, the zero solution is unstable. Here $|c_{ij}(t)|\leqslant c_{ij}$.

(i) $a+b>0$;

(ii) there exists $\varepsilon>0$, so that $\Sigma_{i+j\geqslant 2}c_{ij}|x^i||y^j|<+\infty$ when $|x|<\varepsilon,|y|<\varepsilon$.

Observing the above theory, we may find that Eq. (2) is a functional differential equation, and we can discuss the dynamical characteristic of the origin of Eq. (2) by applying Fact 1. Consequently, we have the following proposition.

Proposition 1: If the parameters satisfy the inequality $|B|<1-A$ (Ia), which means both the value of neuron and the self-excitation dominate the threshold effect, the zero solution of Eq. (2) is globally asymptotically stable, that is, the solution of Eq. (2) damps toward zero as $t\to\infty$.

Proof: According to the assumption, we have $A+B<1$. This leads to the fact that the origin is a unique equilibrium. If $A\geqslant 0$, we take the Lyapunov functional

$$V(\phi)=\phi^2(0)+|B|\int_{-\tau}^{0}\phi^2(\theta)d\theta.$$

Let $\mu(r)=r^2$, and we obtain

$$V(\phi)=\phi^2(0)+|B|\int_{-\tau}^{0}\phi^2(\theta)d\theta\geqslant\phi^2(0)=\mu(|\phi(0)|).$$

Furthermore, we could find a positive definite function $\omega(r)$ satisfies

$$\dot{V}|x_t\leqslant-\omega(|\phi(0)|).$$

Therefore, by Fact 1(1), $\lim_{t\to+\infty}x(t)=0$ as $t\to+\infty$. As the prameter $A<0$, we will use Fact 1(2) to prove that $x(t)$ tends to the largest invariant set $M=\{0\}$ as $t\to+\infty$. These imply that the equilibrium of the neural network Eq. (2) is globally asymptotically stable. (The detailed proof will be given in the Appendix.)

Second, we will analyze the dynamics with a certain and nonignored threshold effect, that is, the parameter region becomes Ib in Fig. 1. The following proposition involves Hopf bifurcation in a neural network model based on fact 2.

Proposition 2: When the parameters in region Ib satisfy $A+B<1,\quad A-B>1$ (which means the system is not only partially but also certainly determined by the effect of the self-inhibitory delay), there exists a critical value τ_c of delay τ, such that the zero solution loses its stability by Hopf bifurcation at τ_c.

Proof: About Eq. (2), the corresponding linear equation is

$$\frac{dx}{dt}=(A-1)x(t)+Bx(t-\tau) \tag{4}$$

and its characteristic equation is $\lambda=A-1+Be^{-\lambda\tau}$.

If $\tau=0$, the characteristic equation becomes $\lambda=A+B-1$. For the assumption $A+B<1$, we know that the characteristic root has a negative real part at $\tau=0$. Thus, the equilibrium is asymptotically stable. There must exist a critical value τ_c such that the characteristic root has a negative part as τ is continuously increased from zero to τ_c, and the characteristic equation has a purely imaginary root while $\tau=\tau_c$. To find this value τ_c, we use the following method.

Letting $\lambda=iv(v\neq 0)$, the characteristic equation turns into

$$iv=A-1+Be^{-iv\tau}=(A-1+B\cos v\tau)-iB\sin v\tau,$$

and we have

$$\begin{cases} A-1+B\cos v\tau=0\\ v+B\sin v\tau=0. \end{cases}$$

This leads to

$$v=\sqrt{B^2-(A-1)^2},\qquad \tau_c=\cos^{-1}[1-A/B]/\sqrt{B^2-(A-1)^2}.$$

Obviously, this satisfies condition (a) in Fact 2. On the other hand, derivation on both side of the characteristic equation $\lambda-Be^{-\lambda\tau}-A+1=0$, with respect to τ, leads to

$$\lambda'(\tau)+Be^{-\lambda\tau}(\tau\lambda'(\tau)+\lambda)=0,$$

that is,

$$\lambda'(\tau)=\frac{-\lambda Be^{-\lambda\tau}}{1+B\tau e^{-\lambda\tau}},$$

and $\lambda'(0)=-B\,\mathrm{Re}\,\lambda(0)\neq 0$. Consequently, condition (b) holds.

According to Fact 2, we know that the zero solution of Eq. (2) produces Hopf bifurcation at τ_c to make the equation have a periodic solution. We can roughly get some properties of the oscillating period when it loses stability. Equation (3) will have a solution $x(t)=e^{iv\tau}$ at (A,B,τ_c), where v is the angle frequency. Therefore, period

$$T=2\pi/v=2\pi/\sqrt{B^2-(A-1)^2}=2\pi\tau_c/\cos^{-1}[1-A/B].$$

Hence, it is easy to see that $2\tau\leqslant T\leqslant 4\tau$. Accordingly, the range of the oscillating period of Eq. (2) is restricted at the unstable point.

Figures 2 and 3 are illustrations of activations of the neuron described by Eq. (2) with the parameters A and B in Region Ib. We take different parameters as follows. The activations of the neuron are shown in Figs. 2 and 3 with the parameters $A=-20.0$, $B=-50.0$, $\tau_c=0.0442$, and $A=1.0$, $B=-100.0$, $\tau_c=0.0157$, respectively. Furthermore, L denotes the largest Lyapunov exponent, which will be explained in the following paragraph. We take the initial condition as $x(\theta)=\phi(\theta)=5.0-5.0\theta/\tau, \theta\in[0,\tau]$.

From the figures shown, we may find that as $\tau<\tau_c$, the origin is asymptotically stable; while τ is close to τ_c, the orbit becomes periodic; when $\tau>\tau_c$, the orbit presents chaotic characteristics after successive periodic changes. This means that the value of a neuron's threshold certainly determine the dynamics of it and the simulation results coincide with our theoretical analysis.

Next, we discuss the situation when the parameter inequality $A+B>1$ holds, that is, in regions IIa and IIb. In these regions, both the feedback and the delay effect are more dominate than the current status and Eq. (2) has three equilibria $x_1=a$, $x_2=0$, $x_3=-a$ with the parameter inequality. Applying Fact 3 above, we can easily conclude the following proposition. (The proof of Proposition 3 is omitted here as it is trivial.)

Proposition 3: If $A+B>1$, the zero solution of Eq. (2) is unstable.

What's more, we want to investigate the characteristic of equilibria at zero as well. First, expand Eq. (2) at $x_1=a$ according to the Taylor formula, and we have

$$\frac{dx}{dt}=-x(t)+A\left\{\frac{e^a-e^{-a}}{e^a+e^{-a}}+\frac{4}{(e^a+e^{-a})^2}[x(t)-a]+\cdots\right\}$$
$$+B\left\{\frac{e^a-e^{-a}}{e^a+e^{-a}}+\frac{4}{(e^a+e^{-a})^2}[x(t-\tau)-a]+\cdots\right\}.$$

Let $z(t)=x(t)-a$, and the variational equation of Eq. (2) converts to

$$\frac{dz}{dt}=-z(t)+\frac{4A}{(e^a+e^{-a})^2}z(t)+\frac{4B}{(e^a+e^{-a})^2}z(t-\tau).$$

Thus, the corresponding characteristic equation is

$$\lambda=-1+\frac{4A}{(e^a+e^{-a})^2}+\frac{4B}{(e^a+e^{-a})^2}e^{-\lambda\tau}.$$

If $\tau=0$, it leads to

$$\lambda=-1+\frac{4A}{(e^a+e^{-a})^2}+\frac{4B}{(e^a+e^{-a})^2}.$$

Let $f(a)=e^{2a}-e^{-2a}-4a$, and we have $f'(a)=2e^{2a}+2e^{-2a}-4=2(e^a-e^{-a})^2>0$. For $f(a)>f(0)=0(a>0)$ and $A+B>1$, the inequality $4(A+B)/(e^a+e^{-a})^2<1$ holds. Therefore, $\lambda<0$ as $\tau=0$. Because λ is continuously dependent on τ, it follows that the characteristic root has a negative real part for small τ, which implies the local asymptotic stability of equilibrium $x_1=a$ as $A+B>1$. Specifically, if $B<0$, we want to find a critical value τ_c' as τ is increased.

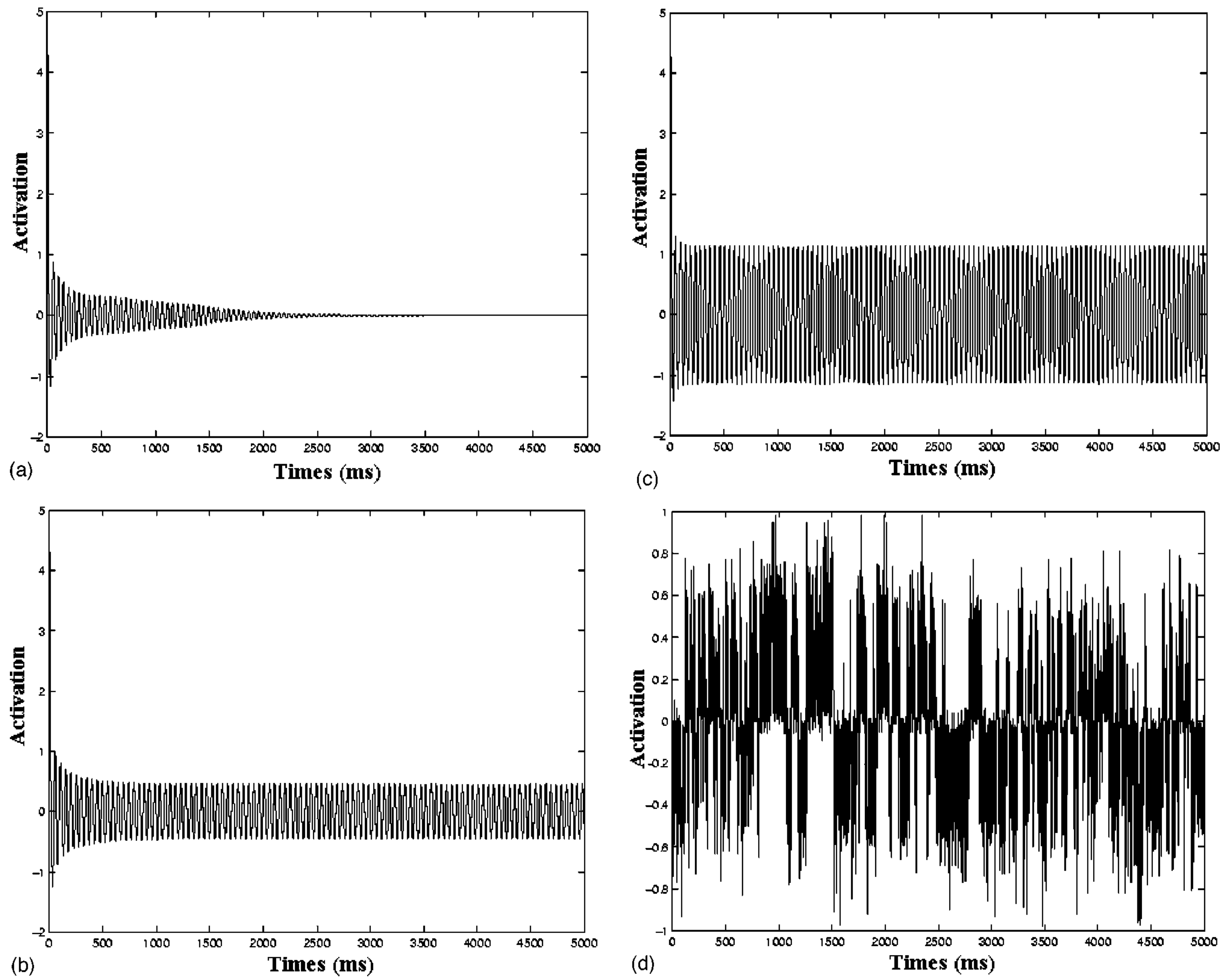

FIG. 2. Dynamics of a scalar delayed neural network with parameters $A=-20.0$, $B=-50.0$, $\tau_c=0.0442$, and the different threshold values noted under each picture. L represents the largest Lyapunov exponent. (a) $\tau=0.042$, (b) $\tau=0.0442$, (c) $\tau=0.058$, (d) $\tau=0.06$, $L=0.2853$.

Letting $\lambda=iv$, the characteristic equation is

$$iv=-1+\frac{4A}{(e^a+e^{-a})^2}+\frac{4B}{(e^a+e^{-a})^2}e^{-iv\tau}$$
$$=-1+\frac{4A}{(e^a+e^{-a})^2}+\frac{4B}{(e^a+e^{-a})^2}(\cos v\tau-i\sin v\tau)$$
$$=-1+\frac{4A}{(e^a+e^{-a})^2}+\frac{4B}{(e^a+e^{-a})^2}$$
$$\times\cos v\tau-i\frac{4B}{(e^a+e^{-a})^2}\sin v\tau.$$

Comparing the imaginary part on both sides, we get

$$v=\sqrt{\frac{16(B^2-A^2)+8A(e^a+e^{-a})^2}{(e^a+e^{-a})^4}-1}\,,$$

therefore,

$$\tau_c'=\cos^{-1}\left[\frac{(e^a+e^{-a})^2-4A}{4B}\right]\Big/\ v.$$

Similarly to the proof of Proposition 2, the equilibrium $x_1=a$ loses its stability by Hopf bifurcation at τ_c'.

By a minor modification of the above discussion, one can easily determine the characteristic equilibrium of $x_3=-a$. In short, for $A+B>1$, $x_2=0$ is unstable and $x_1=a$, $x_3=-a$ are locally asymptotically stable.

In the following paragraph, we are concerned with how the unstable solution of Eq. (2), that is, the current status of a neuron, performs; will it evolve stochastically or regularly or tend toward infinity. By Eq. (2), we know that

$$\left|\frac{dx(t)}{dt}+x(t)\right|=|A\tanh[x(t)]+B\tanh[x(t-\tau)]|$$
$$\leqslant|A||\tanh[x(t)]|+|B||\tanh[x(t-\tau)]|$$
$$\leqslant|A|+|B|;$$

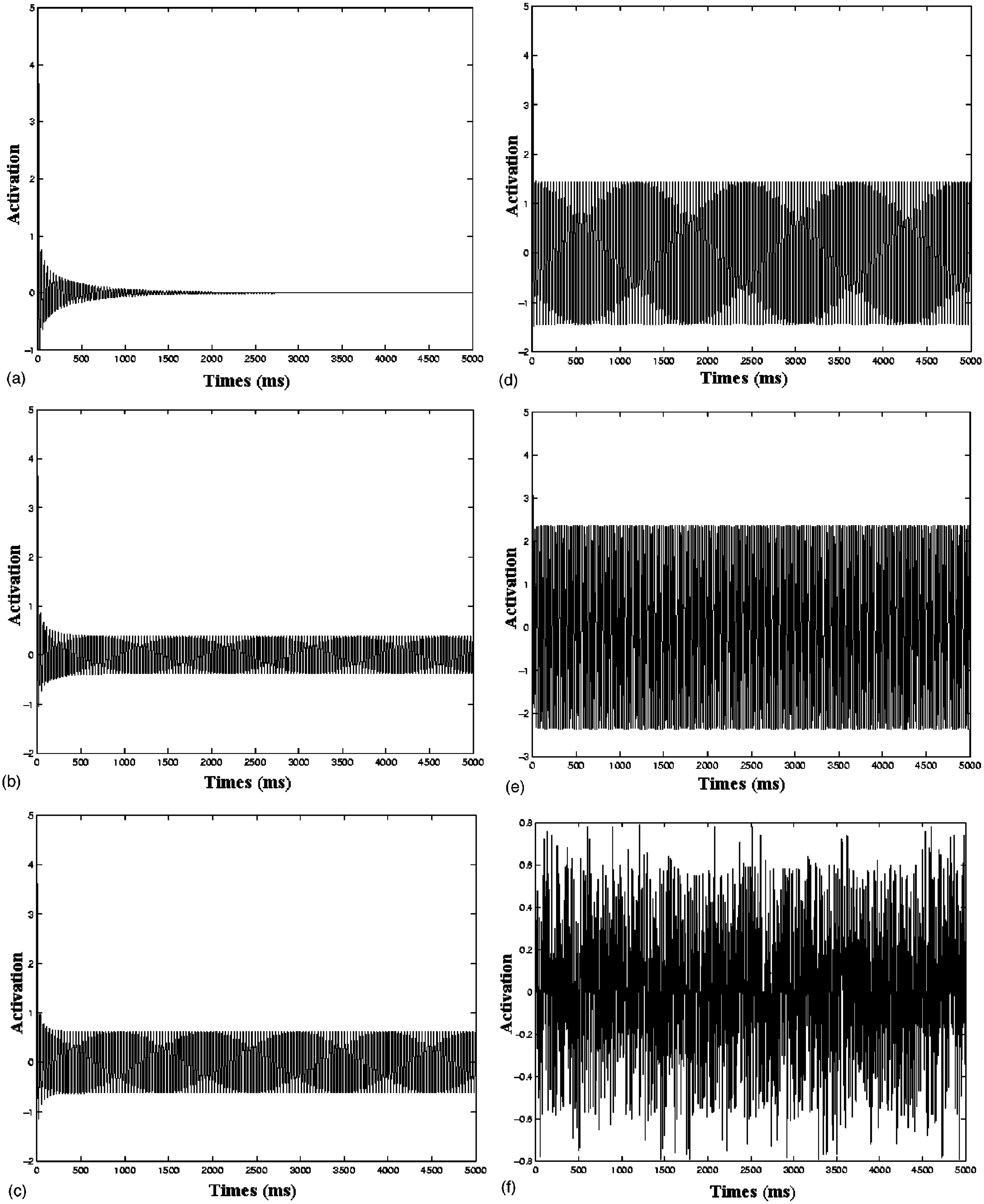

FIG. 3. Regular and complex dynamics of a scalar delayed neural network with parameters $A=1.0$, $B=-100.0$, $\tau_c=0.0157$, and the different threshold values noted under the pictures. L represents the largest Lyapunov exponent. (a) $\tau=0.01$, (b) $\tau=0.015$, (c) $\tau=0.0158$, (d) $\tau=0.022$, (e) $\tau=0.026$, (f) $\tau=0.028$, $L=0.1322$.

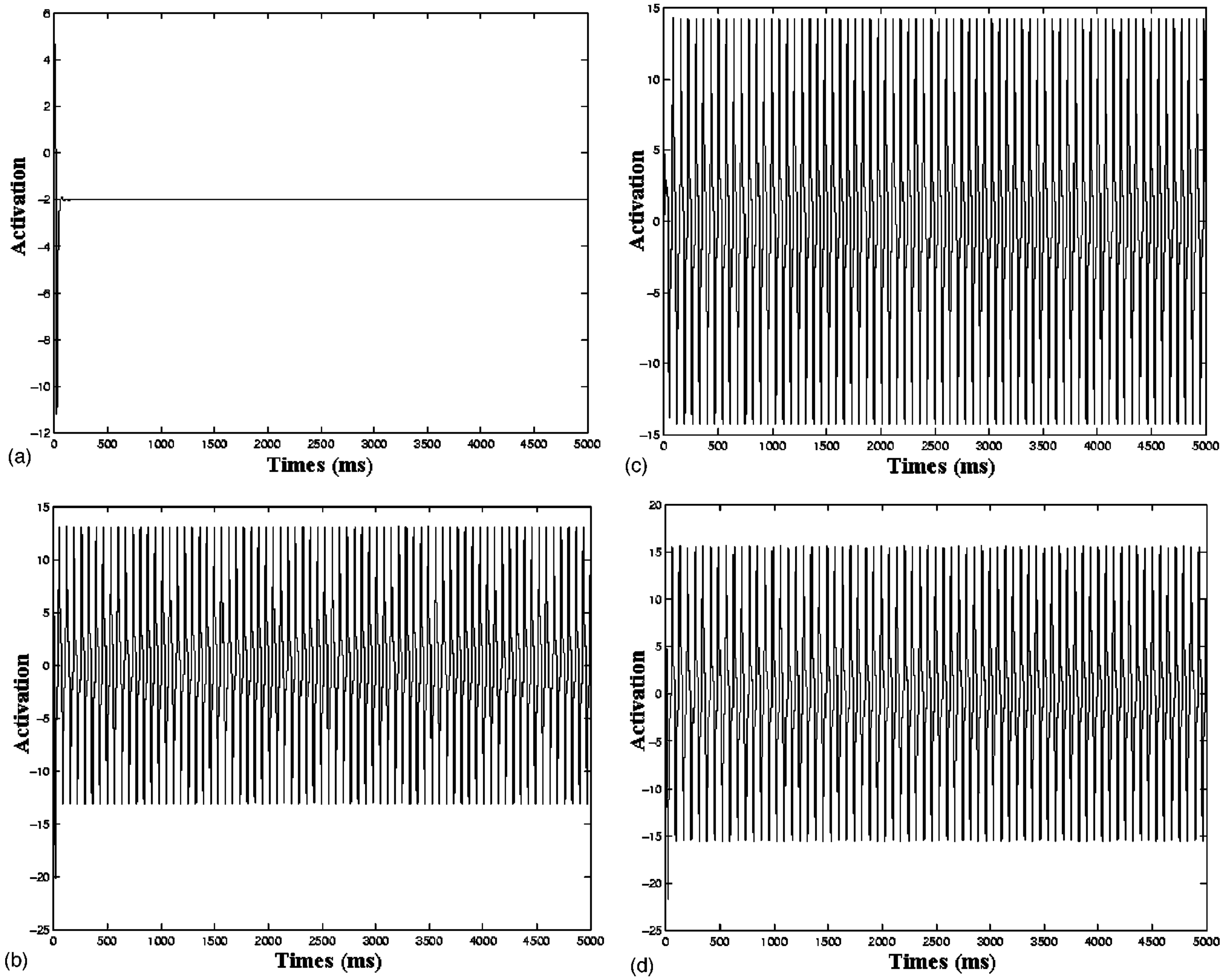

FIG. 4. Asymptotic dynamics and stable oscillations of a scalar delayed neural network system with the particular parameters $A=13.5373$, $B=-11.4627$, and $\tau_c'=2.009$. (a) $\tau=1.0$, (b) $\tau=2.009=\tau_c'$, (c) $\tau=2.20$, (d) $\tau=2.50$.

thus,

$$-(|A|+|B|)\leqslant\frac{dx(t)}{dt}+x(t)\leqslant|A|+|B|.$$

Then we get

$$\int_{t_0}^{t}-(|A|+|B|)e^{s}ds\leqslant xe^{t}-x_0e^{t_0}\leqslant\int_{t_0}^{t}(|A|+|B|)e^{s}ds.$$

Therefore,

$$\begin{cases}-(|A|+|B|)<x\leqslant|A|+|B|+x_0e^{t_0} & (x_0>0)\\ -(|A|+|B|)+x_0e^{t_0}\leqslant x<|A|+|B| & (x_0<0).\end{cases}$$

This leads to the following proposition.

Proposition 4: Every solution of Eq. (2) is bounded.

Our paper is based on the above results. What we considered is that due to the solutions of Eq. (2) all being bounded, the dynamics of the solutions will be very complex as the equilibrium points are unstable. In an ordinary differential equation, a system whose dimension is larger than three will have chaotic dynamics, however, a one-dimensional delay differential equation corresponds to an infinite dimension dynamical system, which means that it is very possible that such a system's dynamics will perform chaotically. Therefore, the activation level of a neuron's performance should be complicated or even chaotic. Motivated by these ideas, we simulate the numerical integral solution of Eq. (2). We study especially the dynamical behavior when delay τ changes near the critical value, and detect the solution trajectories that exhibit prolific dynamics as τ is increased gradually.

Figures 4 and 5 are the numerical simulation of the parameters A, B in region IIb. We take $A=13.5373$, $B=-11.4627$, and $\tau_c'=2.009$. The initial condition is taken as $x(\theta)=\phi(\theta)=5.0-5.0\theta^2/\tau^2$, $\theta\in[0,\tau]$.

Figure 4(a) shows that the activations tend to the fixed point -2. Figures 4(b) and 5 depict the evolution from period to chaos, and the larger the delay τ of a neuron becomes, the more obviously the chaotic phenomenon extends. We zoom out to the activations in the time period 2000–5000 (ms) and 3000–4000 (ms) from those in Fig. 5(b), which can be seen in Figs. 5(c) and 5(d).

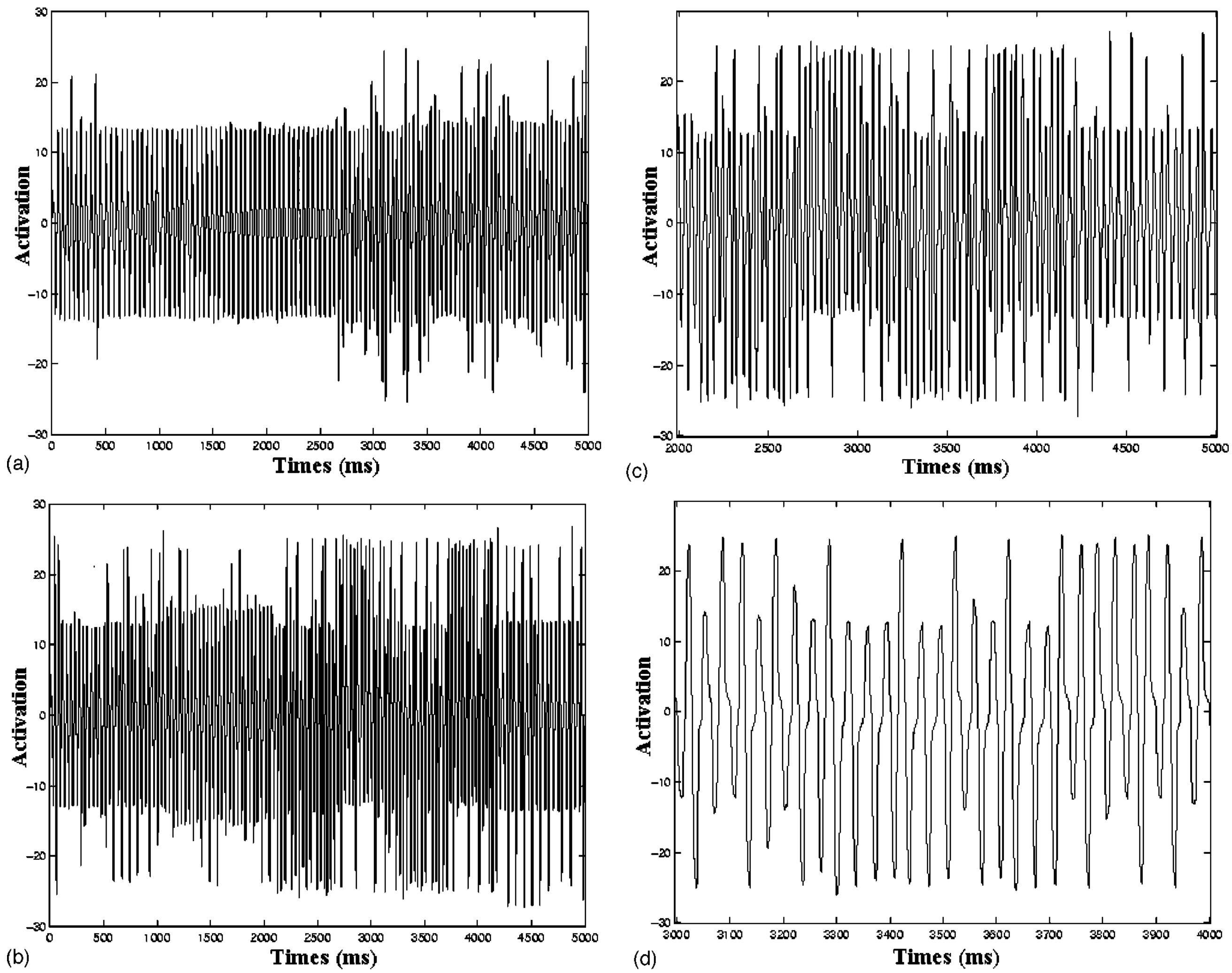

FIG. 5. Chaotic dynamics of scalar delayed neural network systems with larger threshold τ and parameters $A=13.5373$, $B=-11.4627$, $\tau_c'=2.009$, where L is the largest Lyapunov exponent. (a) $\tau=3.0$, $L=1.2531$, (b) $\tau=3.90$, $L=1.4618$. (c) and (d) are the zoomed out pictures from (b) in the time period 2000–5000 (ms) and 3000–4000 (ms), respectively.

Chaotic dynamics are unpredictable because the evolutional time is of long enough duration to make it impossible to depict the orbital evolution by using the analytic solution of Eq. (2). However, we are trying to give an effective theoretical criterion to ensure the existence of chaos; nevertheless, it is still in process. Numerical simulation is then often used as an auxiliary tool, designed for this, and the research on chaos is usually called ''the science of the computer era.'' We also use the largest Lyapunov exponent [14]. This exponent is widely used in nonlinear science as a measurement index. It can be found [15] that a positive largest Lyapunov exponent indicates the appearance of chaos in a bounded system. The following computation simulation results affirm our conjecture.

As $A=-20.0$, $B=-50.0$, and $\tau=0.06$, the largest Lyapunov exponent is 0.2853; the largest Lyapunov exponent becomes 0.1322 with the parameters $A=1.0$, $B=-100.0$ and $\tau=0.028$, while with $A=13.5373$, $B=-11.4627$, and $\tau=2.8$, the largest Lyapunov exponent is equal to 1.1794. It is evident that the activations, limited in a bounded region, and split exponentially, show roughly the emergence of chaos.

In Ref. [4], the chaos of such a neuron in the neuronic system, described by Eq. (2), is never mentioned; however, we observe the chaotic orbit with the aid of a computer. Our results further strengthen the evidence demonstrating the presence of chaos.

III. DYNAMICS OF THE NEURAL NETWORKS MODEL DESCRIBED IN A *n*-DIMENSIONAL DELAY FUNCTIONAL DIFFERENTIAL EQUATION WITH SYMMETRICAL WEIGHT MATRIX

In Ref. [1], Marcus and Westervelt considered the following equations with delay:

$$\dot{u}_i(t)=-u_i(t)+\sum_{j=1}^{n} J_{ij}f(u_j(t-\tau)) \quad (i=1,2,\cdots,n). \tag{5}$$

The model is originally from the Hopfield system, and where J_{ij} represents the connected weight between the ith neuron and the jth neuron, and the second term on the right-hand side of Eq. (5) represents the feedback from all neurons to the ith neuron itself. However, Marcus and Westervelt add time delay to the connection of terms of this system such that it becomes more realistic for the real neural networks. Equation (5)'s linear equations are

$$\dot{u}_i(t) = -u_i(t) + \sum_{j=1}^{n} \beta J_{ij} u_j(t-\tau), \tag{6}$$

where β is the derivative of $f(u)$ at 0.

Suppose J_{ij} is symmetrical, and that it can be transformed into a diagonal matrix. Therefore, the above n-dimensional equations will turn into the following n one-dimensional equations:

$$\dot{x}_i(t) = -x_i(t) + \beta\lambda_i x_i(t-\tau), \tag{7}$$

where $\lambda_i (i=1,2,\ldots,n)$ is the eigenvalue of J_{ij}.

We considered the following equation, similar to Eq. (5):

$$\dot{u}_i(t) = R_i u_i(t) + \sum_{j=1}^{n} W_{ij} f[u_j(t-\tau)], \quad (i=1,2,\ldots,n), \tag{8}$$

where $R_i<0$ represents the resistance of each neuron, and W_{ij} represents the connected weight between the ith neuron and the jth neuron. However, Eq. (5) is a special form from Eq. (8). Equation (8)'s linear equations turn out to be

$$\dot{u}_i(t) = R_i u_i(t) + \sum_{j=1}^{n} f'(0) W_{ij} u_j(t-\tau), \quad (i=1,2,\ldots,n). \tag{9}$$

In the following paragraph, we show some results in an n-dimensional system, which is analogous to that of the scalar neuron system. For convenience of discussion, we take $f(u)=\tanh(u)$, which is a popular function in a neural network system, and $\beta=1$. By the direct use of some facts in Appendix B, we have the following results. (We omit the proof here to avoid repetition.)

Proposition 5.

(1) If all eigenvalues λ_i of weight matrix W satisfy $|\lambda_i|<|R_i|$ $(i=1,2,\ldots,n)$ (which means the delay effect is not dominate in the evolution of a neural network system), then the zero solution of Eq. (9) is asymptotically stable, independent of the time delay τ.

(2) The equation corresponding to x_i satisfying $\lambda_i=-R_i$ has the property that its zero solution is stable for any time delay τ.

(3) If all eigenvalues $\lambda_i<-R_i$ $(i=1,2,\ldots,n)$, and there exists at least one λ_j $(j\in\{1,2,\ldots,n\})$ satisfying $\lambda_j<R_j$, and then there exists a $\tau_c>0$ such that as $\tau\in(0,\tau_c)$, the zero solution of Eq. (9) is asymptotically stable; as $\tau>\tau_c$, the stability is lost. Here,

$$\tau_c = \min_j \frac{1}{\sqrt{\lambda_j^2-R_j^2}} \cos^{-1}\left(\frac{R_j}{\sqrt{\lambda_j^2-R_j^2}}\right)$$

$$(\lambda_j<R_j, j\in\{1,2,\ldots,n\}).$$

Note: When $\tau\in(0,\tau_c)$, the zero solution of Eq. (9) is asymptotically stable. When $\tau=\tau_c$, because several components satisfying $\lambda_j<R_j$ produce Hoof bifurcation, the whole system tends to a periodic solution while the other components tend to zero; when $\tau>\tau_c$, the components satisfying $\lambda_j<R_j$ have complex chaotic dynamics after successive Hoof bifurcation, thus the total n-dimension system is chaotic as long as there is one chaotic component. In short, the threshold value is the key to the different dynamics when the delayed self-inhibition is considerably large, that is, $\lambda_i<R_i$.

Proposition 6. If there exists at least one λ_i ($i\in\{1,2,\ldots,n\}$) satisfying $\lambda_i>-R_i$, then the zero solution of Eq. (9) is unstable for any $\tau>0$.

Remark. By this proposition, once there is one chaotic component of a neural network system, the n-dimensional system should be chaotic. As a result, it can be seen that as long as there is at least one $\lambda_i>-R_i$, then for any $\tau>0$, it will be possible to make the n-dimensional system chaotic owing to the chaotic dynamics of certain components. Comparatively, requiring all $|\lambda_i|<|R_i|$ to make the zero solution asymptotically stable is obviously strictly limited; however, complex behavior such as chaos is relatively easy to demonstrate.

Numerical simulation results

Now, by applying the above theory, we take a special case that is a two-dimensional system as

$$R_1=-21, \quad R_2=1, \quad W_{11}=-50, \quad W_{22}=-100,$$

$$W_{12}=W_{21}=0, \quad [\text{Figs. 6(a)–6(b)}]$$

$$R_1=-21, \quad R_2=-3, \quad W_{11}=-50, \quad W_{22}=-4,$$

$$W_{12}=W_{21}=0, \quad [\text{Fig. 6(c)}].$$

Thus, we have the following numerical simulation results.

Figure 6(a) shows the phase space of components u_1 and u_2 with the time delay $\tau=0.01$ [$\tau_c=\min(0.0452,0.0157)=0.0157$]. They show the convergence of two components to zero, which means that the status of the two neurons is asymptotically stable.

Figure 6(b) shows the evolutions of components u_1 and u_2, respectively, as the time delay $\tau=0.06$. We can see that the orbits perform chaotically. And the corresponding orbits in phase space are also shown in the third picture of Fig. 6(b). We also get a limit cycle, shown in Fig. 6(c), of the neural networks' activations as the parameters are taken as the above explanation; therefore, the status of the two neurons tends toward regular oscillation with time.

IV. CONCLUSION

All of the obtained criteria and propositions are strictly proved by the Lyapunov functional approach and Hopf Bifurcation. These results are reinforced by the computer simulations. The conjecture of the existence of the chaos in neural network system is analyzed in detail above. These factors might play an important role in the design of neural net-

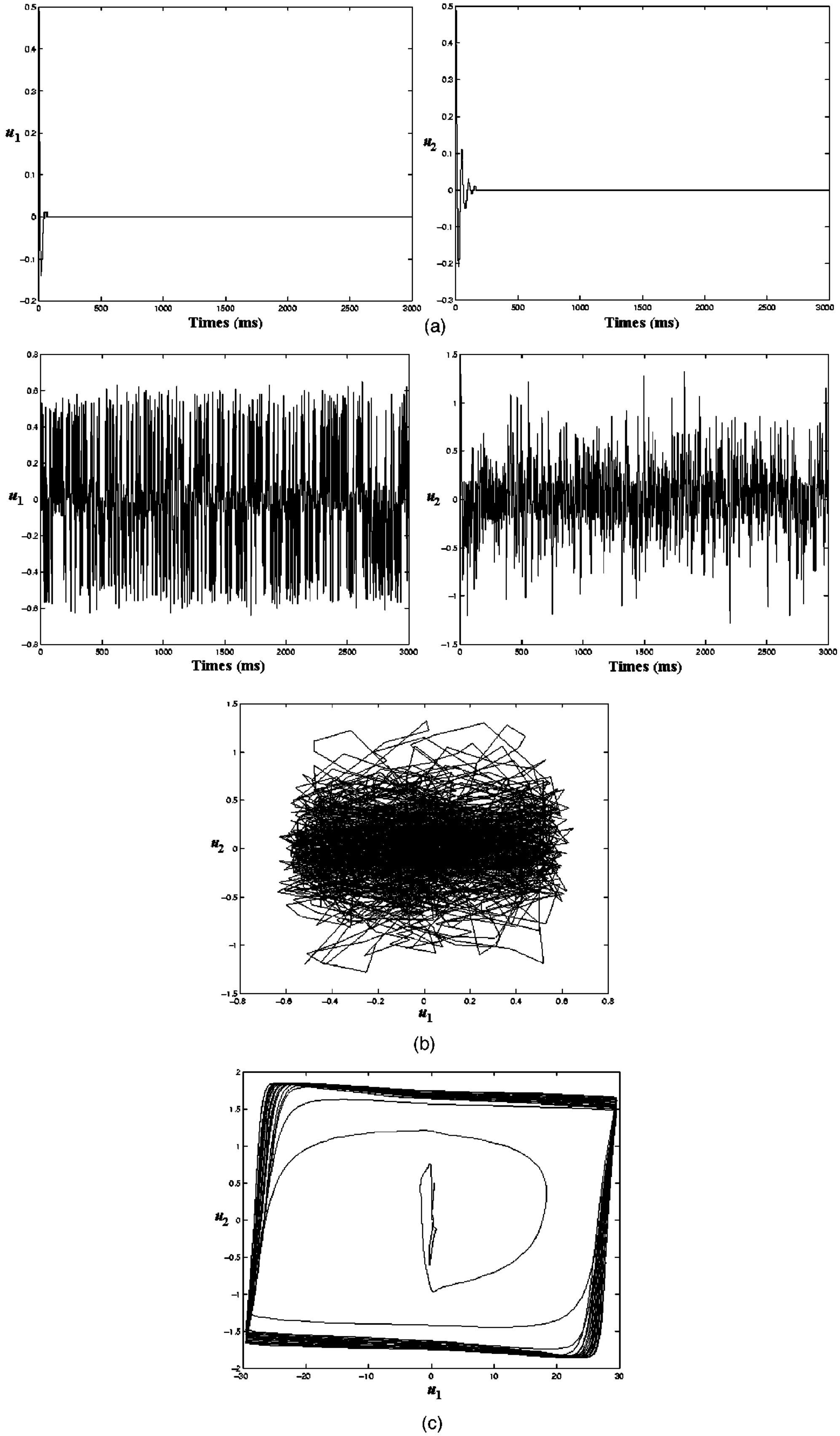

FIG. 6. Dynamics of two-dimensional neural networks with $R_1=-21$, $R_2=1$, $W_{11}=-50$, $W_{22}=-100$, $W_{12}=W_{21}=0$, [(a), (b)] and $R_1=-21$, $R_2=-3$, $W_{11}=-50$, $W_{22}=-4$, $W_{12}=W_{21}=0$. (a) $\tau=0.01$, (b) $\tau=0.06$, (c) $\tau=4.0$.

works. However, the theoretical proofs about the existence of chaos in neural network models with delay, considering the stochasticity of real neural networks, will be given in future studies.

ACKNOWLEDGMENTS

This work was supported by both the National Natural Foundation of China and the Natural Science Foundation of Shanghai.

APPENDIX A

The details of proof Proposition 1 are given as follows. We take

$$V(\phi)=\phi^2(0)+|B|\int_{-\tau}^{0}\phi^2(\theta)d\theta.$$

Hence,

$$\begin{aligned}\dot{V}|_{x_t}&=2x(t)\dot{x}(t)+|B|x^2-|B|x^2(t-\tau)\\&=-2x^2(t)+2Ax(t)\tanh[x(t)]+2Bx(t)\tanh[x(t-\tau)]\\&\quad+|B|x^2(t)-|B|x^2(t-\tau),\end{aligned}$$

We know that

$$\tanh[x(t)]=x(t)\tanh'(\xi_1),$$

$$\tanh[x(t-\tau)]=x(t-\tau)\tanh'(\xi_2),$$

where ξ_1 is the value between 0 and $x(t)$, ξ_2 is the value between 0 and $x(t-\tau)$, and $0\leqslant\tanh'(s)\leqslant 1$ for any s. As the parameter satisfies $A\geqslant 0$, we have

$$\begin{aligned}\dot{V}|_{x_t}&\leqslant-2x^2(t)+2Ax^2(t)+2|B||x(t)||x(t-\tau)|\\&\quad+|B|x^2(t)-|B|x^2(t-\tau)\\&=-2(1-A-|B|)x^2(t)-|B|[|x(t)|-|x(t-\tau)|]^2\\&\leqslant-2(1-A-|B|)x^2(t).\end{aligned}$$

Thus, we take $\omega(r)=-2(1-A-|B|)r^2$, which is positive definite. And the origin of the neural network with delay is globally asymptotically stable as $A\geqslant 0$.

On the other hand, if the parameter satisfies $A<0$, we take

$$\begin{aligned}V(\phi)&=\phi^2(0)+\int_{-\tau}^{0}\phi^2(\theta)\mathrm{d}\theta+A\int_{-t}^{0}[\phi(\theta)]^2\mathrm{d}\theta\\&\quad+A\int_{-t}^{-\tau}[\phi(\theta)]^2\mathrm{d}\theta.\end{aligned}$$

Therefore,

$$\begin{aligned}\dot{V}|_{x_t}&\leqslant-2x^2(t)+2Ax^2(t)\tanh'(\xi_1)+2|B||x(t)||x(t-\tau)|\\&\quad+x^2(t)-x^2(t-\tau)+Ax^2(t)+Ax^2(t-\tau)\\&=2Ax^2(t)\tanh'(\xi_1)+(A-1)x^2(t)\\&\quad+2|B||x(t)||x(t-\tau)|]+(A-1)x^2(t-\tau)\end{aligned}$$

For $\Delta=4|B|^2-4(A-1)^2<0$, it leads to

$$\dot{V}|_{x_t}\leqslant 2Ax^2(t)\tanh'(\xi_1)\leqslant 0.$$

Thus, $S=\{\phi|\dot{V}(\phi)=0\}=\{\phi|\phi(0)=\phi(-\tau)=0\}$, and it is obvious that its largest invariant set $M=\{0\}$. By the second criterion of Fact 1, we can prove that the origin of the neural network with delay is globally asymptotically stable as $A<0$. Above all, we complete the proof of Proposition 1.

APPENDIX B

The following facts will be useful to the proof of the proposition in Sec. III.

Fact 4 [13]: Regarding the equation $\dot{x}(t)=ax(t)+bx(t-\tau)$, if coefficients a and b satisfy $a+b<0$, and there exists $\Delta=\Delta(a,b)>0$, so that when $\tau\in[0,\Delta(a,b)]$, the zero solution is asymptotically stable. Here,

$$\Delta(a,b)=\begin{cases}+\infty & (|a|>|b|)\\ \dfrac{1}{\sqrt{b^2-a^2}}\cos^{-1}\left(\dfrac{a}{\sqrt{b^2-a^2}}\right) & (|a|<|b|).\end{cases}$$

Fact 5 [13]: If the coefficients a and b of equation $\dot{x}(t)=ax(t)+bx(t-\tau)$ satisfy $a+b>0$, then the zero solution is unstable for any $\tau>0$.

Fact 6 [13]: Regarding equation $\dot{x}(t)=ax(t)-ax(t-\tau)$, there exist $\Delta=\Delta(a)>0$, so that when $\tau\in(0,\Delta)$, the zero solution is stable. Specifically, when $a\leqslant 0$, $\Delta=+\infty$.

[1] C. M. Marcus and R. M. Westervelt, Pays. Red A **39**, 347 (1989).
[2] K. Pakdaman, C. P. Malta, C. Grotta-Ragazzo, and J.-F. Vibert, Neural Comput. **9**, 319 (1997).
[3] K. Gopalsamy and I. K. C. Lung, IEEE Trans. Neural Netw. **8**, 341 (1997).
[4] K. Pakdaman and C. P. Malta, IEEE Trans. Neural Netw. **9**, 231 (1998).
[5] K. Gopalsamy and I. K. C. Lung, Physica D **76**, 344 (1996).
[6] L. Olien and J. Bélair, Physica D **102**, 349 (1997).
[7] J. Wei and S. Ruan, Physica D **130**, 255 (1999).
[8] Hui Fang and Jibin Li, Phys. Rev. E **61**, 4212 (2000).
[9] K. Pakdaman, C. P. Malta, C. Grotta-Ragazzo, O. Arino, and J.-F. Vibert, Phys. Rev. E **55**, 3234 (1997).
[10] N. Brunel and V. Hakim, Neural Comput. **11**, 1621 (1999).
[11] E. R. Caianiello and A. Be Lucas, Kybernetik **3**, 33 (1966).

[12] J. K. Hale, *Theory of Functional Differential Equations* (Springer-Verlag, Berlin, 1977).

[13] Qin Yuanxun, Liu Yongqing, Wang Lian, and Zheng Zuxiu, *The Stability of Dynamical System with Delay* (Science, New York, 1998).

[14] M. T. Rosenstein, J. J. Collies and C. J. DeLuca, Physica D **65**, 117 (1993).

[15] J. Gleick, *Chaos: Making a New Science* (Penguin, New York, 1988).